NOT IF, BUT HOW

FIVE

Towards robust portfolios

Hierarchical risk parity: persistent diversification uncovered by graph theory and machine learning

FIVE

We believe in numbers.

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In recent years, machine learning (ML) techniques have found their way into almost all areas of practical applications, and many financial experts started to apply these methods to financial markets and asset allocation. Asset allocation is one core factor in determining the risk/return profile for an investment portfolio. The hierarchical risk parity allocation (HRP) provides a new modern portfolio construction technique which improves the robustness and diversification of a portfolio in many cases and thus contributes to refined portfolio performance.

At Munich Re, we launched the FIVE Robust Multi-Asset Index (VROBUST) applying the HRP methodology on a broadly diversified portfolio consisting of global equity indices, government bonds and commodities. The HRP portfolio exhibits superior risk-adjusted returns and shows better results regarding drawdown metrics when compared to most other common allocation methods.

Classical portfolio allocation techniques typically are either based on simple approaches which do not incorporate the correlation between the assets, e.g. equal weights (EW), fixed weights (FW) or inverse volatility (naive risk parity, NRP) or the allocation techniques are based on more complex approaches depending on forecasting the covariance matrix and calculating its inversion. Examples of these more complex alternatives are minimum variance (MV), most diversified portfolio (MDP) and equally-weighted risk contribution (ERC). By their nature, the simple approaches cannot incorporate different market environments representing changing correlation structures as they simply ignore correlations. The complex methods optimise a risk-adjusted performance target function sensitive to an inversion of the predicted future covariance matrix. Non-stationary return time series due to changing market behaviour lead to uncertainties in the estimators for this future covariance matrix and thus to a lack of robustness of the resulting weights, causing an unnecessary high turnover.

The hierarchical risk parity (HRP) approach developed by de Prado (2016) strikes a new path by incorporating graph theory and machine learning techniques to derive a more robust allocation method. The algorithm takes advantage of the correlation structure without being dependent on the inversion of the covariance matrix. The comprehensible and straightforward construction of HRP makes it also an ideal candidate not only for classical long-only portfolios but also for alternative beta strategies which use long-short positions for implementing their investment idea.

What problems do traditional allocation approaches face?

The so-called Modern Portfolio Theory (MPT) goes back to Nobel prize laureate Harry M. Markowitz and today is the scientific foundation for many asset managers to compose their investment portfolios. MPT addresses the benefits of diversification and acknowledges the fact that assets need to be assessed in a portfolio context.

Choosing an adequate combination of the selected assets can make a fundamental difference. Mathematically speaking, Markowitz formulates a quadratic optimisation problem trying to find a portfolio with the best ratio of return to risk. However, financial market data tends to be complex data with fluctuating statistical features. It turned out that MPT is too sensitive to the stationarity of return time series so that even small forecasting and estimation errors without structural breaks in market behaviour can lead to dramatically different efficient frontiers (Michaud, 1989; Broadie, 1993). The uncertainties associated with the estimation and prediction of the efficient frontier lead to unstable and unpredictable out-of-sample results. This is known as the

Markowitz's curse: the higher the correlations and the more diversification is needed, the more pronounced the failure of MPT.

Even more, returns can rarely be forecasted with sufficient accuracy, and as a consequence, many practitioners and model developers have dropped return forecasting altogether.

Since the minimum variance approach is very susceptible to inaccuracies, it becomes evident that the outcome is at high risk of adverse market developments. Minor changes in the covariance matrix, divergent realisation or inaccuracies with the matrix inversion can have distinct and erratic effects (cf. figure 1).

The Markowitz's curse

	Return (e)*	**Volatility** (e)*	**Weight**
ASSET 1	10%	10%	33.3%
ASSET 2	10%	10%	33.3%
ASSET 3	10%	10%	33.3%

Base case: assume a portfolio with 3 assets and a correlation of 90% between each of the asset pairs. As all have the same return/risk attributes, the algorithm weights are equally distributed – in line with intuition.

	Return (e)	**Volatility** (e)	**Weight**
ASSET 1	10%	10%	50%
ASSET 2	10%	10%	50%
ASSET 3	10%	10.5%	0%

In portfolios with highly correlated assets, small changes in expected volatilites can produce very different weights. In the above case, a slightly increased risk for asset 3 is sufficient to set its weight to 0%.

	Return (e)	**Volatility** (e)	**Weight**
ASSET 1	10%	10%	0%
ASSET 2	10%	10%	0%
ASSET 3	11%	10%	100%

A small increase in the expected return is enough to remove asset 1 and 2 entirely from the portfolio. The "optimal" solution now consists of one asset. Again, small estimation errors can have severe effects on the allocation.

*(e): expected

Figure 1

Examples of an optimisation for the tangency portfolio. The solution provides the asset weights which are supposed to realise the highest performance quality. Estimation errors might eliminate the diversification benefits. MPT allocation is very sensitive to its input, especially in high correlation cases.

Source: Munich Re

The quadratic optimisation in the Markowitz approach enlarges small estimation errors; this is called the error maximisation property, giving rise to risk-based asset allocation approaches like risk budgeting (with equally-weighted risk contribution (ERC) as special case) or the maximum diversified portfolio approach. However, numerous studies show that quadratic optimisers in general produce unreliable solutions. Many of the best known quadratic optimisers underperform the naive equal weight (EW) allocation out-of-sample.

The equally-weighted risk contribution (ERC) approach became famous in the years following the Dotcom Bubble. Due to diversification effects disappearing during the Global Financial Crisis (2007–2009) as assets suddenly tended to move in sync, ERC missed to outperform during that period.

Using a minimum spanning tree, the weights of a minimum variance optimisation are illustrated in figure 2. The size of the nodes is proportional to the weights.

Weights of a minimum variance optimisation

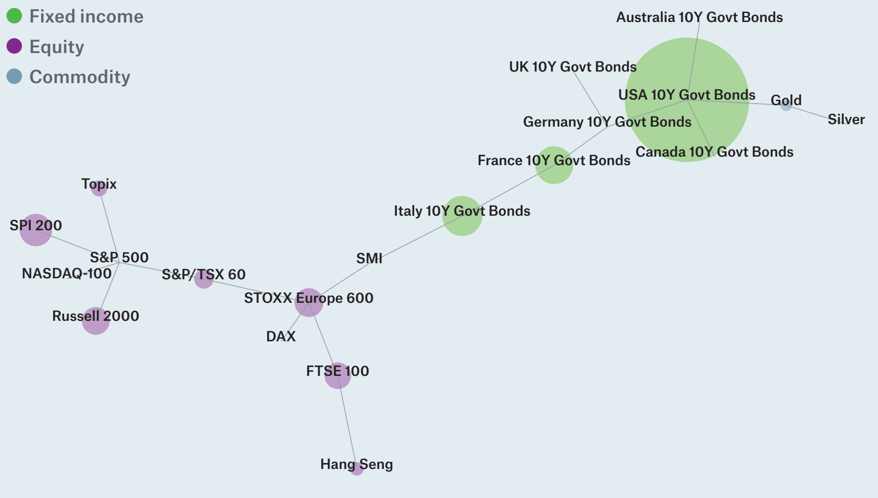

Figure 2

The minimum variance approach tends to allocate high weights on assets with a low variance and a low overall correlation. Many constituents are obtaining a zero weight and are removed from the portfolio.

Source: Munich Re

The minimum spanning tree is created by a filtering algorithm which successively removes less important links to reveal a pronounced structure of the complex portfolio information.

Illustration of the minimum spanning tree algorithm

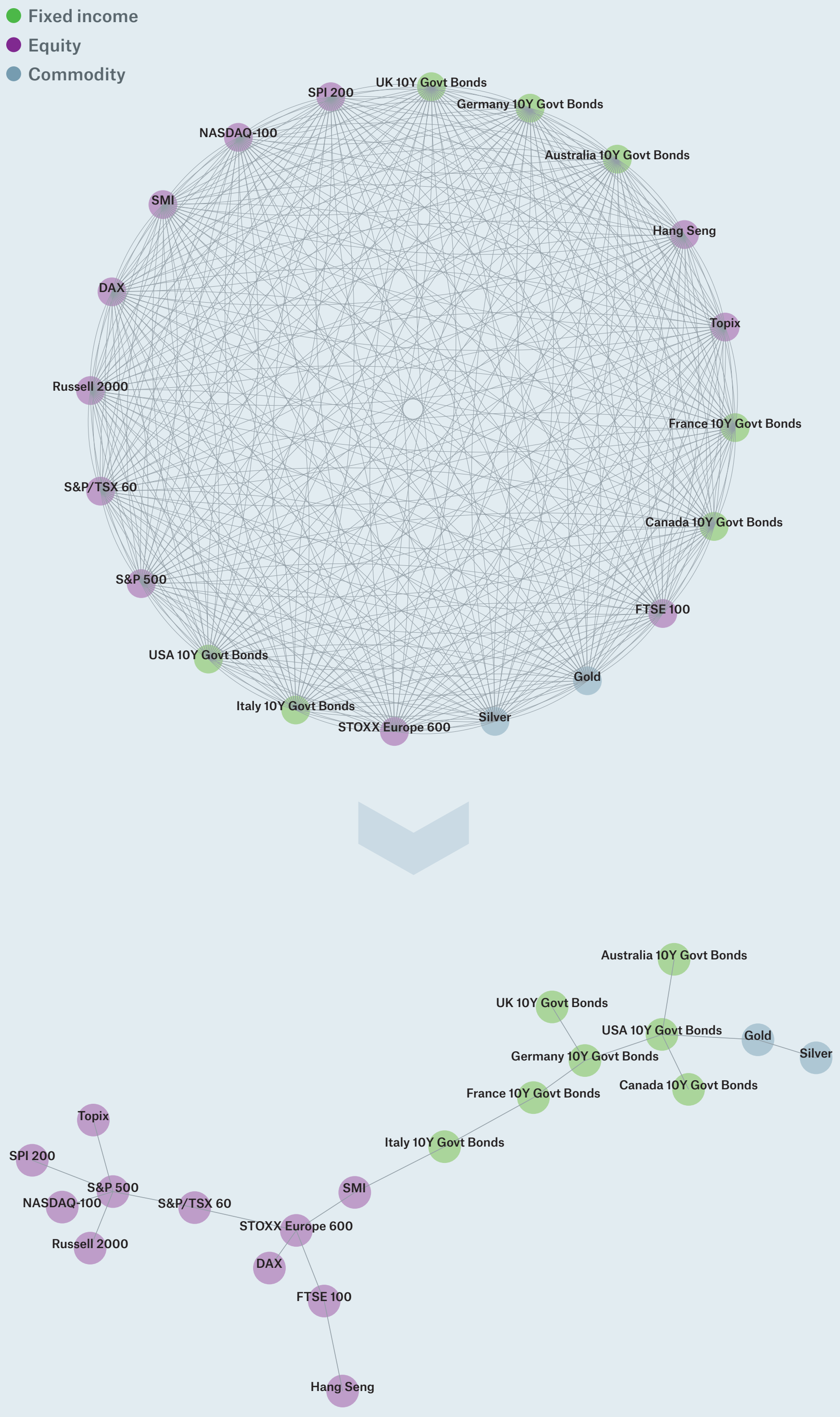

Figure 3

Classical allocation techniques often take into account the correlations between all asset pairs in the portfolio. This is illustrated in the upper part of the figure, where a very large and fully connected graph of correlation dependencies can be seen: each asset is linked to all the other assets. The minimum spanning tree identifies the most important links, and drops less relevant ones.

Source: Munich Re

The hierarchical risk parity approach

The HRP approach introduced by de Prado (2016) defines a new robust portfolio diversification technique. It is based on the fundamental idea that complex systems such as financial markets show a hierarchical structure and that assets can be grouped into clusters with similar behaviour and risk profile.

The design persuades to avoid the pitfalls of traditional allocation methods such as inverting the covariance matrix or not taking into account the correlation structure at all. The clusters tend to be stable over time, leading to less fluctuation in the allocation and a robust out-of-sample outperformance compared to other allocation methodologies.

The robustness of the approach was shown in Jaeger et al. (2020) using a block-bootstrapping resampling method: 100,000 bootstrapped scenarios were generated from the empirical dataset and analysed in depth. HRP was found to show better risk-adjusted returns and lower maximum drawdowns compared to NRP and ERC.

The first step of the HRP approach is to use a hierarchical clustering algorithm to group assets with a similar risk profile together. Hierarchical clustering is mathematically similar to the minimum spanning tree. It reveals a nested cluster structure that is often visualised using a so-called dendrogram. This is done by defining a distance metric based on the pairwise correlation of the assets. Applying a hierarchical cluster algorithm provides a hierarchical clustering tree grouping the assets into specific risk clusters uncovering the pronounced structure of complex market dependencies.

De Prado (2016) uses the single-linkage clustering algorithm for building the clusters. Notably, the single-linkage clustering is an equivalent representation of the minimum spanning tree. Since this method suffers from chaining and tends to deliver sparse clusters, we modify the original HRP approach and instead apply Ward's method (Ward, 1963) which generally forms more homogenous clusters.

The dendrogram in figure 4 visualises the grouping of the assets in our portfolio.

Hierarchical clustering

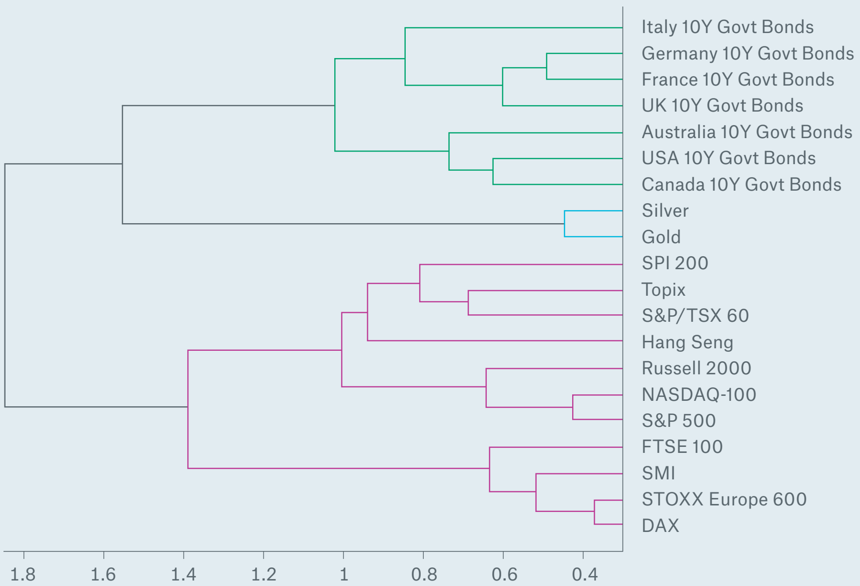

Figure 4

Assets with a similar risk profile are grouped together. Even regional dependencies become evident, e.g. S&P500, NASDAQ-100 and Russel 2000 or FTSE 100, SMI, STOXX Europe 600 and DAX are forming clusters.

Source: Munich Re

In a second step the cluster tree is used to quasi-diagonalise the covariance matrix. Figure 5 illustrates the quasi-diagonalisation for the correlation matrix. High correlations are marked in red and low correlations are marked in green.

Quasi-diagonalisation of the correlation matrix

1.0	0.4	(0.0)	(0.1)	(0.0)	0.4	0.3	(0.2)	0.5	(0.0)	0.1	0.2	0.3	0.2	0.3	0.4	(0.2)	0.4	0.4	0.0
0.4	1.0	0.1	0.0	0.1	0.5	0.5	(0.0)	0.5	0.2	0.2	0.1	0.5	0.4	0.3	0.6	0.0	0.9	0.4	0.1
(0.0)	0.1	1.0	0.2	0.4	(0.0)	(0.4)	0.5	(0.2)	0.3	0.2	(0.0)	(0.3)	0.1	(0.2)	(0.0)	0.2	0.0	(0.3)	0.8
(0.1)	0.0	0.2	1.0	0.6	(0.0)	0.1	0.6	(0.2)	0.5	0.5	0.4	0.1	(0.1)	0.1	(0.1)	0.5	(0.1)	0.0	0.2
(0.0)	0.1	0.4	0.6	1.0	(0.0)	(0.1)	0.7	(0.1)	0.8	0.8	0.5	(0.2)	(0.1)	(0.2)	(0.1)	0.6	(0.0)	(0.2)	0.3
0.4	0.5	(0.0)	(0.0)	(0.0)	1.0	0.5	(0.1)	0.6	(0.1)	0.0	0.0	0.5	0.3	0.4	0.6	0.0	0.6	0.5	0.1
0.3	0.5	(0.4)	0.1	(0.1)	0.5	1.0	(0.2)	0.5	(0.1)	0.1	0.2	0.7	0.4	0.6	0.5	(0.1)	0.5	0.9	(0.2)
(0.2)	(0.0)	0.5	0.6	0.7	(0.1)	(0.2)	1.0	(0.2)	0.6	0.5	0.2	(0.3)	(0.2)	(0.2)	(0.2)	0.7	(0.1)	(0.3)	0.4
0.5	0.5	(0.2)	(0.2)	(0.1)	0.6	0.5	(0.2)	1.0	(0.1)	0.0	0.0	0.5	0.4	0.3	0.5	(0.1)	0.6	0.5	(0.2)
(0.0)	0.2	0.3	0.5	0.8	(0.1)	(0.1)	0.6	(0.1)	1.0	0.7	0.4	(0.1)	(0.1)	(0.0)	(0.2)	0.5	(0.0)	(0.2)	0.2
0.1	0.2	0.2	0.5	0.8	0.0	0.1	0.5	0.0	0.7	1.0	0.7	(0.0)	(0.1)	(0.1)	(0.0)	0.5	0.1	(0.0)	0.2
0.2	0.1	(0.0)	0.4	0.5	0.0	0.2	0.2	0.0	0.4	0.7	1.0	0.0	(0.1)	0.0	(0.0)	0.3	0.0	0.1	(0.0)
0.3	0.5	(0.3)	0.1	(0.2)	0.5	0.7	(0.3)	0.5	(0.1)	(0.0)	0.0	1.0	0.4	0.7	0.5	(0.2)	0.6	0.8	(0.2)
0.2	0.4	0.1	(0.1)	(0.1)	0.3	0.4	(0.2)	0.4	(0.1)	(0.1)	(0.1)	0.4	1.0	0.5	0.3	(0.1)	0.4	0.5	0.1
0.3	0.3	(0.2)	0.1	(0.2)	0.4	0.6	(0.2)	0.3	(0.0)	(0.1)	0.0	0.7	0.5	1.0	0.4	(0.1)	0.4	0.8	(0.0)
0.4	0.6	(0.0)	(0.1)	(0.1)	0.6	0.5	(0.2)	0.5	(0.2)	(0.0)	(0.0)	0.5	0.3	0.4	1.0	(0.2)	0.8	0.5	0.0
(0.2)	0.0	0.2	0.5	0.6	0.0	(0.1)	0.7	(0.1)	0.5	0.5	0.3	(0.2)	(0.1)	(0.1)	(0.2)	1.0	(0.1)	(0.1)	0.2
0.4	0.9	0.0	(0.1)	(0.0)	0.6	0.5	(0.1)	0.6	(0.0)	0.1	0.0	0.6	0.4	0.4	0.8	(0.1)	1.0	0.6	0.1
0.4	0.4	(0.3)	0.0	(0.2)	0.5	0.9	(0.3)	0.5	(0.2)	(0.0)	0.1	0.8	0.5	0.8	0.5	(0.1)	0.6	1.0	(0.2)
0.0	0.1	0.8	0.2	0.3	0.1	(0.2)	0.4	(0.2)	0.2	0.2	(0.0)	(0.2)	0.1	(0.0)	0.0	0.2	0.1	(0.2)	1.0

high correlation

low correlation

1.0	0.7	0.6	0.8	0.4	0.4	0.3	0.5	0.3	0.4	0.3	(0.2)	(0.0)	0.1	(0.1)	(0.2)	0.0	(0.0)	(0.1)	(0.2)
0.7	1.0	0.7	0.8	0.5	0.6	0.5	0.4	0.3	0.5	0.5	(0.3)	(0.2)	0.1	(0.2)	(0.3)	0.0	(0.1)	(0.0)	(0.2)
0.6	0.7	1.0	0.9	0.5	0.5	0.5	0.4	0.3	0.5	0.5	(0.4)	(0.2)	0.1	(0.1)	(0.2)	0.2	(0.1)	0.1	(0.1)
0.8	0.8	0.9	1.0	0.5	0.6	0.4	0.5	0.4	0.5	0.5	(0.3)	(0.2)	0.0	(0.1)	(0.3)	0.1	(0.2)	(0.0)	(0.2)
0.4	0.5	0.5	0.5	1.0	0.8	0.6	0.3	0.4	0.6	0.5	(0.0)	0.0	(0.1)	(0.2)	(0.2)	(0.0)	(0.2)	(0.0)	(0.1)
0.4	0.6	0.5	0.6	0.8	1.0	0.9	0.4	0.4	0.6	0.6	0.0	0.1	(0.1)	(0.1)	(0.1)	0.0	(0.0)	0.1	(0.0)
0.3	0.5	0.5	0.4	0.6	0.9	1.0	0.4	0.4	0.5	0.5	0.1	0.1	0.0	0.0	(0.0)	0.1	0.2	0.2	0.1
0.5	0.4	0.4	0.5	0.3	0.4	0.4	1.0	0.2	0.3	0.4	0.1	0.1	(0.1)	(0.1)	(0.2)	(0.1)	(0.1)	(0.1)	(0.1)
0.3	0.3	0.3	0.4	0.4	0.4	0.4	0.2	1.0	0.4	0.5	(0.0)	0.0	(0.1)	(0.2)	(0.2)	0.2	(0.0)	0.1	(0.0)
0.4	0.5	0.5	0.5	0.6	0.6	0.5	0.3	0.4	1.0	0.6	(0.0)	0.1	(0.0)	0.0	(0.1)	0.0	(0.1)	0.0	(0.0)
0.3	0.5	0.5	0.5	0.5	0.6	0.5	0.4	0.5	0.6	1.0	(0.2)	(0.2)	(0.2)	(0.1)	(0.2)	0.0	(0.1)	0.0	(0.1)
(0.2)	(0.3)	(0.4)	(0.3)	(0.0)	0.0	0.1	0.1	(0.0)	(0.0)	(0.2)	1.0	0.8	0.2	0.2	0.5	(0.0)	0.3	0.2	0.4
(0.0)	(0.2)	(0.2)	(0.2)	0.0	0.1	0.1	0.1	0.0	0.1	(0.2)	0.8	1.0	0.2	0.2	0.4	(0.0)	0.2	0.2	0.3
0.1	0.1	0.1	0.0	(0.1)	(0.1)	0.0	(0.1)	(0.1)	(0.0)	(0.2)	0.2	0.2	1.0	0.5	0.6	0.4	0.5	0.5	0.6
(0.1)	(0.2)	(0.1)	(0.1)	(0.2)	(0.1)	0.0	(0.1)	(0.2)	0.0	(0.1)	0.2	0.2	0.5	1.0	0.7	0.3	0.5	0.5	0.6
(0.2)	(0.3)	(0.2)	(0.3)	(0.2)	(0.1)	(0.0)	(0.2)	(0.2)	(0.1)	(0.2)	0.5	0.4	0.6	0.7	1.0	0.2	0.6	0.5	0.7
0.0	0.0	0.2	0.1	(0.0)	0.0	0.1	(0.1)	0.2	0.0	0.0	(0.0)	(0.0)	0.4	0.3	0.2	1.0	0.4	0.7	0.5
(0.0)	(0.1)	(0.1)	(0.2)	(0.2)	(0.0)	0.2	(0.1)	(0.0)	(0.1)	(0.1)	0.3	0.2	0.5	0.5	0.6	0.4	1.0	0.7	0.8
(0.1)	(0.0)	0.1	(0.0)	(0.0)	0.1	0.2	(0.1)	0.1	0.0	0.0	0.2	0.2	0.5	0.5	0.5	0.7	0.7	1.0	0.8
(0.2)	(0.2)	(0.1)	(0.2)	(0.1)	(0.0)	0.1	(0.1)	(0.0)	(0.0)	(0.1)	0.4	0.3	0.6	0.6	0.7	0.5	0.8	0.8	1.0

Figure 5

The correlation matrix is reordered using the order of the hierarchical clustering forming blocks of high correlation.

Source: Munich Re

The final step of the algorithm then calculates the weight for each of the assets using a recursive bi-sectioning procedure of the reordered covariance matrix. We start at the top of the tree and with a weight of 1 for each asset.

Then we divide the assets into two equal subsets ("bi-sectioning") and rescale the weights by multiplying each weight with the inverse proportion of its subsets variance. Both subsets are divided again, and the weights are rescaled respectively. Recursively the final weights are thereby derived. The details can be found in de Prado (2016). Figure 6 illustrates the recursive algorithm:

Recursive allocation

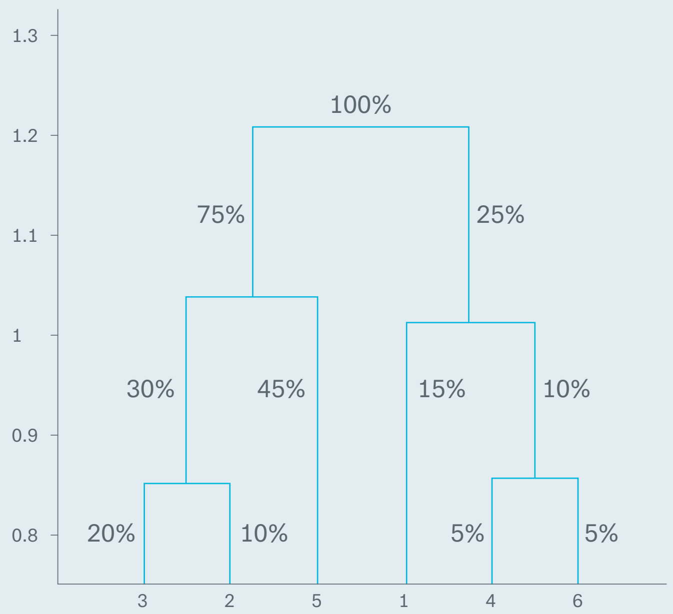

Figure 6

Weights are distributed optimally following an inverse-variance allocation. The recursive algorithm ensures that the weights are positive and sum up to 1 on each split level.

Source: Munich Re

One important result of this construction is that the HRP allocation avoids high risk concentration. This is due to the hierarchical clustering which groups highly correlated assets together and low correlated assets are arranged far away. Nevertheless, the method is taking into account dynamic changes in the correlation matrix. The construction method apparently ensures stability in the allocation. Minor changes in the covariance matrix only lead to minor changes in the final weight allocation.

The HRP diversifies well across asset classes, regions, dynamically changing asset clusters and even across single positions/futures. It takes a harmonised approach across capital weights and risk contributions of assets and grouped assets. As will be seen later, HRP is not focused entirely on a single diversification measure. Instead, it takes a balanced position in a wide range of many different diversification measures. It seems that the approach benefits from this behaviour since there does not exists one single best diversification measure.

What are the results of such a rich diversification approach? The wide range of diversification properties covered by HRP makes the strategy less prone to a range of shock scenarios. The likelihood of hidden portfolio concentrations, which could potentially lead to unexpectedly large losses, is reduced. Shocks in financial markets can originate from single assets but also from groups of asset which lead to collective or systemic shocks. HRP positions the portfolio in a way to mitigate those shocks.

The FIVE Robust Multi-Asset Index (VROBUST[1])

We applied the hierarchical risk parity approach to create the FIVE Robust Multi-Asset Index (VROBUST). The long-only asset universe is comprised of 20 well-diversified futures markets, covering equity index, government bond and commodity markets. The index is an excess return[2] index; it is rebalanced monthly on the first business day of the month and scaled to target a realised index volatility of 5% per annum. The index includes a fee of 25bps and a realistic implementation of transaction costs.

Investment portfolio

Constituent	Asset class	Currency	Constituent	Asset class	Currency
Australia 10Y Govt Bonds	Fixed Income	AUD	STOXX Europe 600	Equities	EUR
Canada 10Y Govt Bonds	Fixed Income	CAD	FTSE 100	Equities	GBP
France 10Y Govt Bonds	Fixed Income	EUR	Hang Seng	Equities	HKD
Germany 10Y Govt Bonds	Fixed Income	EUR	NASDAQ-100	Equities	USD
Italy 10Y Govt Bonds	Fixed Income	EUR	Russell 2000	Equities	USD
UK 10Y Govt Bonds	Fixed Income	GBP	S&P 500	Equities	USD
USA 10Y Govt Bonds	Fixed Income	USD	S&P/TSX 60	Equities	CAD
Gold	Commodities	USD	SMI	Equities	CHF
Silver	Commodities	USD	SPI 200	Equities	AUD
DAX	Equities	EUR	Topix	Equities	JPY

Table 1
Source: Munich Re

Figure 7 illustrates the index performance[3] over time and shows a comparison to traditional allocation methods such as equal weight (EW), naive risk parity (NRP), equally-weighted risk contribution (ERC) and minimum variance (MV).

Comparison of historical index performance

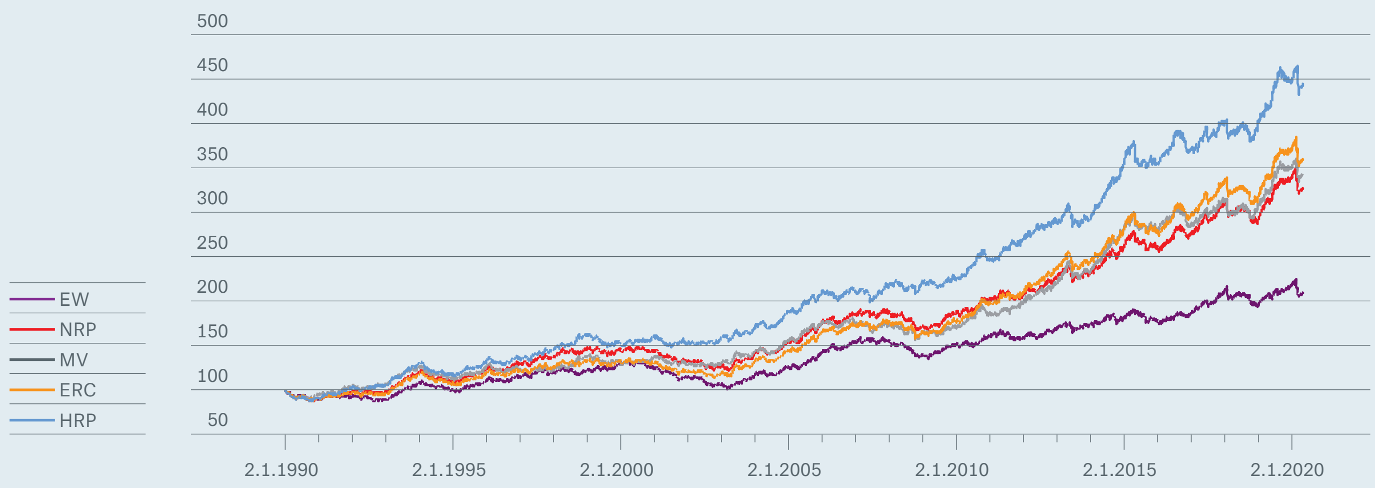

Figure 7
Source: Munich Re

[1] ISIN: DE000SL0ASR0, Reuters ticker: .VROBUST, Bloomberg ticker: VROBUST <Index>
[2] The USD-denominated excess return version has been chosen for this article, as it allows for better comparability and strategy quality analysis. Furthermore, as the portfolio components are FX-hedged, the choice of the index base currency has minor impact on the index excess return performance.
[3] Past performance is no indication of future performance.

Table 2 summarises selected key performance metrics for the different allocation methods.

Performance Metric

	EW	NRP	MV	ERC	HRP
Compound annual growth rate	2.5%	4.0%	4.1%	4.3%	5.0%
Volatility	4.9%	4.9%	5.1%	5.0%	5.0%
Sharpe ratio[4]	0.51	0.81	0.80	0.86	1.00
Maximum drawdown	−23.5%	−17.9%	−13.5%	−15.6%	−12.2%
Sortino ratio[5]	0.81	1.30	1.31	1.39	1.64
Calmar ratio[6]	0.11	0.23	0.31	0.28	0.42
Mean leverage	70%	106%	142%	120%	128%

Table 2
Selected key performance metrics (based on daily returns and annualised if applicable).
Source: Munich Re

It is well known that the performance of an investment strategy is often dominated by the extreme outcomes of a few trading days. HRP can perform comparably well in extreme market environments: during the crises following the Dotcom Bubble and the collapse of Lehman Brothers or during the recent pandemic crisis of Covid-19.

[4] The Sharpe ratio measures the excess performance compared to a risk-free asset of an investment adjusting for its risk.
[5] The Sortino ratio is a modification of the Sharpe ratio. It relates the excess performance to its downside deviation.
[6] The Calmar ratio is defined as the annualized average discrete rate of return divided by the maximum drawdown.

Case Studies

We perform three case studies for a deeper understanding how the HRP allocation performs in different crises. We decided to study the Great Bond Massacre in 1994, the Global Financial Crisis 2007–2009 and the ongoing Coronavirus Pandemic in 2020. These 3 crises are very different in their nature, source and effects and thus help us to understand the benefits of the HRP allocation principle.

Great Bond Massacre 1994

The 1994 bond market crisis or Great Bond Massacre was a sharp sell-off in the bond markets starting in Japan and the US in January 1994 due to fast increased interest rates under FED Chair Alan Greenspan. The crisis spread through almost all developed markets. In 1994 the Federal Reserve increased interest rates from 3% to 5.5% by the end of the year.

1 January 1994– 31 January 1995	EW	NRP	MV	ERC	HRP
Commodities	–1.1%	–1.1%	–2.6%	–2.2%	–1.6%
Equities	–4.6%	–3.1%	0.6%	–2.8%	–1.4%
Fixed Income	–2.8%	–5.2%	–8.2%	–5.4%	–7.3%
Total	**–8.6%**	**–9.4%**	**–10.3%**	**–10.5%**	**–10.3%**

Table 3

Asset-class specific performance attribution (discrete returns).

Source: Munich Re

Given the high bond allocation in the HRP portfolio it is not surprising that HRP is one of the worst performing portfolios in this scenario. But remarkably all portfolios perform comparably poor and ERC which is much acclaimed for its good diversification records the sharpest decline.

The Global Financial Crisis 2007–2009

The financial crisis led to extreme drawdowns in equity and commodity markets but to falling yields and therefore increasing bond prices. We choose the period from the top to the bottom of the S&P 500 index for our analysis.

9 October 2007– 9 March 2009	EW	NRP	MV	ERC	HRP
Commodities	0.2%	0.2%	0.1%	0.2%	0.3%
Equities	–14.5%	–16.0%	–12.4%	–14.2%	–6.8%
Fixed Income	1.4%	5.5%	8.0%	8.0%	13.1%
Total	**–12.9%**	**–10.3%**	**–4.3%**	**–6.1%**	**6.5%**

Table 4

Source: Munich Re

We observe that the HRP method manages to avoid the steep drawdowns in equity and commodity markets and made significant gains in the fixed income segment. All other methods including MV accumulate high losses predominant in equity markets and cannot compensate this with gains in fixed income.

Coronavirus Pandemic 2020

The ongoing coronavirus pandemic which started end of 2019 led to one of the fastest downturns in equity markets ever recorded. On top, one of the biggest oil price shocks ever lead to immense losses in shorter term WTI and Brent oil futures. Even gold showed a slump of more than 10% from the top. Government bond markets experienced a roller coaster ride: for example, 10 year US rates first dropped from 1.87% on 1 January 2020 to a low of 0.32% on 9 March, just to recover to 1.2% within one week and afterwards stabilising around 60 bps. Again HRP manages to avoid deeper drawdowns and shows the best performance of the considered methods.

1 January 2020–30 April 2020	EW	NRP	MV	ERC	HRP
Commodities	–1.9%	–2.1%	–1.3%	–1.9%	–0.2%
Equities	–2.6%	–2.2%	–6.4%	–2.3%	–4.7%
Fixed Income	0.5%	1.2%	5.7%	1.6%	4.2%
Total	**–4.0%**	**–3.0%**	**–2.0%**	**–2.6%**	**–0.8%**

Table 5
Source: Munich Re

HRP systematically diversifies which leads to robustness. At the same time, it participates in broader market recoveries due to its harmonised approach on diversification. It basically and systematically harvests the diversification return, which leads to the following performance tableau covering all possible (calendar annual) investment horizons starting in 2001.

Return Triangle

Investment \ Divestment	2019	2018	2017	2016	2015	2014	2013	2012	2011	2010	2009	2008	2007	2006	2005	2004	2003	2002	2001	2000	1999	1998	1997	1996	1995	1994	1993	1992	1991
2019	12.2%																												
2018	5.6%	-0.5%																											
2017	6.2%	3.3%	7.3%																										
2016	5.8%	3.8%	6.0%	4.8%																									
2015	4.7%	2.9%	4.0%	2.4%	0.2%																								
2014	7.3%	6.3%	8.1%	8.4%	10.2%	21.3%																							
2013	6.3%	5.3%	6.5%	6.3%	6.8%	10.4%	0.4%																						
2012	6.4%	5.6%	6.6%	6.5%	6.9%	9.3%	3.8%	7.3%																					
2011	6.8%	6.1%	7.1%	7.1%	7.5%	9.5%	5.8%	8.6%	9.9%																				
2010	7.2%	6.6%	7.6%	7.6%	8.1%	9.7%	7.0%	9.3%	10.4%	10.8%																			
2009	6.5%	5.9%	6.7%	6.6%	6.9%	8.0%	5.6%	6.9%	6.8%	5.2%	-0.1%																		
2008	6.2%	5.7%	6.3%	6.2%	6.4%	7.3%	5.1%	6.1%	5.8%	4.4%	1.4%	2.9%																	
2007	5.9%	5.4%	6.0%	5.8%	6.0%	6.7%	4.8%	5.5%	5.2%	4.0%	1.9%	2.9%	2.9%																
2006	5.6%	5.1%	5.6%	5.4%	5.5%	6.1%	4.3%	4.9%	4.5%	3.4%	1.6%	2.2%	1.9%	0.9%															
2005	5.9%	5.5%	6.0%	5.9%	5.9%	6.5%	5.0%	5.6%	5.4%	4.6%	3.5%	4.4%	4.9%	5.9%	11.0%														
2004	6.2%	5.8%	6.2%	6.2%	6.3%	6.9%	5.5%	6.1%	6.0%	5.4%	4.5%	5.5%	6.1%	7.2%	10.5%	10.0%													
2003	6.2%	5.9%	6.3%	6.2%	6.3%	6.9%	5.7%	6.2%	6.1%	5.6%	4.9%	5.7%	6.3%	7.2%	9.4%	8.5%	7.1%												
2002	6.1%	5.8%	6.2%	6.1%	6.2%	6.7%	5.6%	6.1%	5.9%	5.5%	4.9%	5.6%	6.1%	6.7%	8.2%	7.2%	5.9%	4.7%											
2001	5.5%	5.2%	5.5%	5.4%	5.5%	5.9%	4.7%	5.1%	4.9%	4.4%	3.8%	4.2%	4.4%	4.7%	5.5%	4.1%	2.2%	-0.1%	-4.7%										
2000	5.5%	5.1%	5.5%	5.4%	5.4%	5.7%	4.7%	5.1%	4.9%	4.4%	3.8%	4.3%	4.4%	4.6%	5.3%	4.2%	2.8%	1.3%	-0.3%	4.3%									
1999	4.9%	4.6%	4.9%	4.7%	4.7%	5.0%	4.0%	4.3%	4.0%	3.6%	2.9%	3.2%	3.3%	3.3%	3.7%	2.5%	1.0%	-0.4%	-2.1%	-0.7%	-5.5%								
1998	5.2%	4.9%	5.2%	5.1%	5.1%	5.4%	4.5%	4.8%	4.6%	4.2%	3.6%	4.0%	4.1%	4.2%	4.7%	3.8%	2.8%	1.9%	1.3%	3.3%	2.8%	11.9%							
1997	5.3%	5.0%	5.2%	5.1%	5.2%	5.5%	4.6%	4.9%	4.7%	4.3%	3.8%	4.2%	4.3%	4.4%	4.8%	4.1%	3.3%	2.7%	2.2%	4.1%	4.0%	9.1%	6.3%						
1996	5.2%	4.9%	5.1%	5.0%	5.0%	5.3%	4.5%	4.7%	4.6%	4.2%	3.8%	4.1%	4.2%	4.3%	4.6%	3.9%	3.2%	2.7%	2.3%	3.8%	3.7%	6.9%	4.5%	2.7%					
1995	5.5%	5.2%	5.4%	5.4%	5.4%	5.7%	4.9%	5.2%	5.0%	4.7%	4.3%	4.7%	4.8%	5.0%	5.3%	4.8%	4.2%	3.9%	3.7%	5.2%	5.4%	8.3%	7.1%	7.6%	12.7%				
1994	4.8%	4.5%	4.8%	4.7%	4.6%	4.9%	4.1%	4.3%	4.1%	3.8%	3.4%	3.6%	3.7%	3.7%	4.0%	3.4%	2.7%	2.2%	1.9%	2.9%	2.7%	4.4%	2.6%	1.4%	0.8%	-9.9%			
1993	5.4%	5.2%	5.4%	5.4%	5.4%	5.6%	4.9%	5.2%	5.0%	4.8%	4.4%	4.7%	4.9%	5.0%	5.3%	4.9%	4.4%	4.1%	4.1%	5.2%	5.3%	7.3%	6.4%	6.4%	7.7%	5.2%	22.9%		
1992	5.3%	5.1%	5.3%	5.2%	5.3%	5.5%	4.8%	5.0%	4.9%	4.7%	4.3%	4.6%	4.7%	4.8%	5.1%	4.7%	4.3%	4.0%	3.9%	4.9%	5.0%	6.6%	5.7%	5.6%	6.4%	4.4%	12.3%	2.7%	
1991	5.6%	5.4%	5.6%	5.5%	5.6%	5.8%	5.2%	5.4%	5.3%	5.1%	4.8%	5.1%	5.2%	5.4%	5.7%	5.3%	4.9%	4.7%	4.8%	5.7%	5.9%	7.4%	6.8%	6.9%	7.7%	6.6%	12.6%	7.9%	13.4%

Table 6

Source: Munich Re

The stability of the return triangle confirms robust performance over time. This also means the result of an investment is rather independent of market timing which historically is hard to find in investment practice. Drawdowns are less intense and rather short compared to peers.

NRP and ERC tend to outperform HRP slightly in bull markets due to a higher concentration in equity exposure. However, in the long run, it pays off for HRP to mitigate its vulnerability to tail events that typically arise due to concentrated risks, thus decrease the negative effects of tail events and participate in up-trending markets by broader diversification.

To better understand the key aspects of the favourable diversification of HRP figure 8 visualises a representative allocation of a MV, HRP and NRP portfolio.

HRP is well balanced (1): notional weight perspective

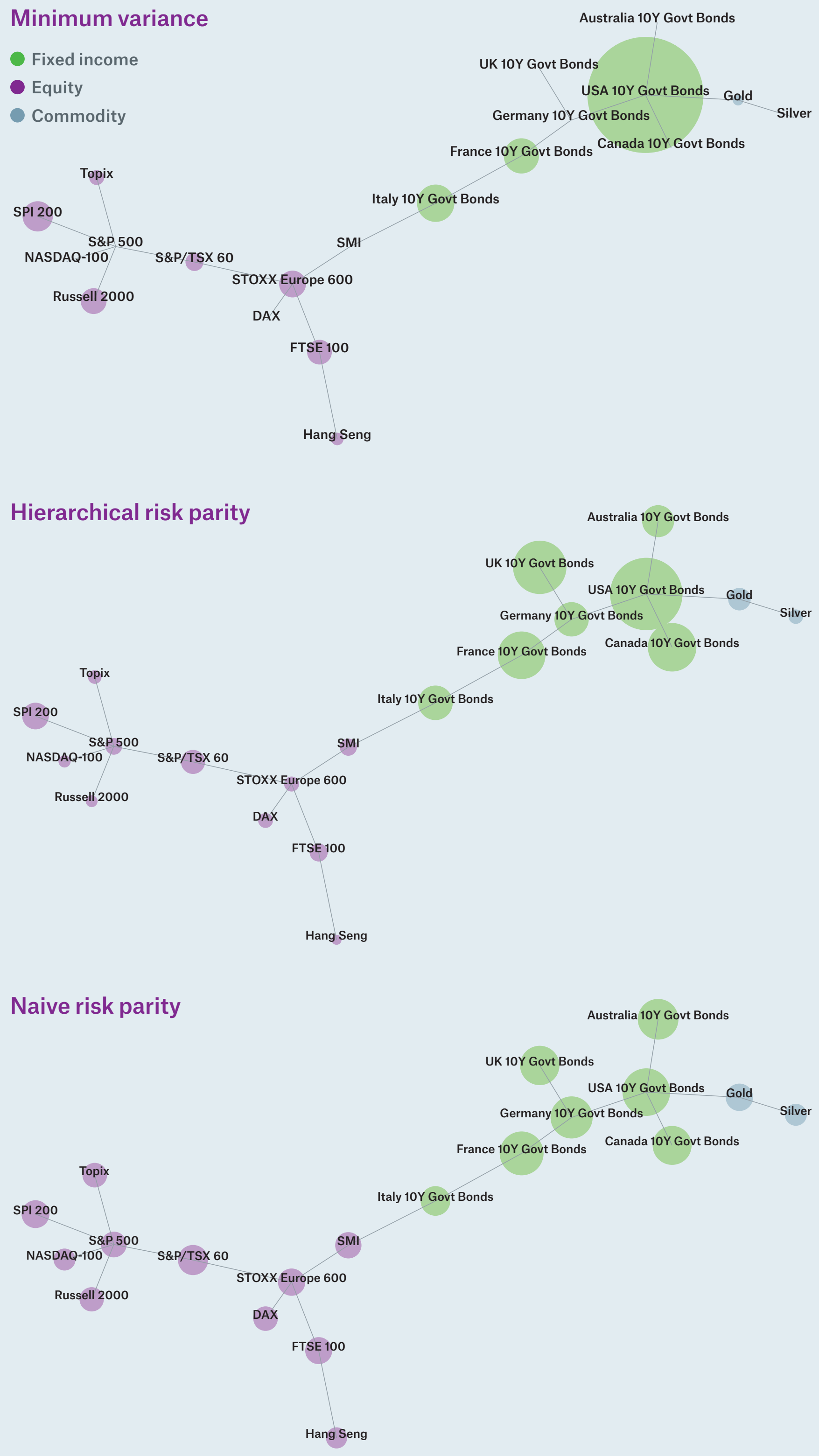

Figure 8

HRP is in many cases able to cure shortcomings of classical allocation techniques, and provides truly balanced portfolios: while considering all components, it tends to avoid risk concentration.

Source: Munich Re

We observe that the HRP portfolio tends to find a balance between more extreme portfolios. Compared to the minimum variance portfolio, it distributes the weights more even between all assets. There are no assets with zero weight and the weight concentrations are less distinct.

To deeply understand where the risk of a portfolio is stemming from, it is important to consider the risk contribution of each asset to the overall portfolio risk (figure 9).

HRP is well balanced (2): risk weight perspective

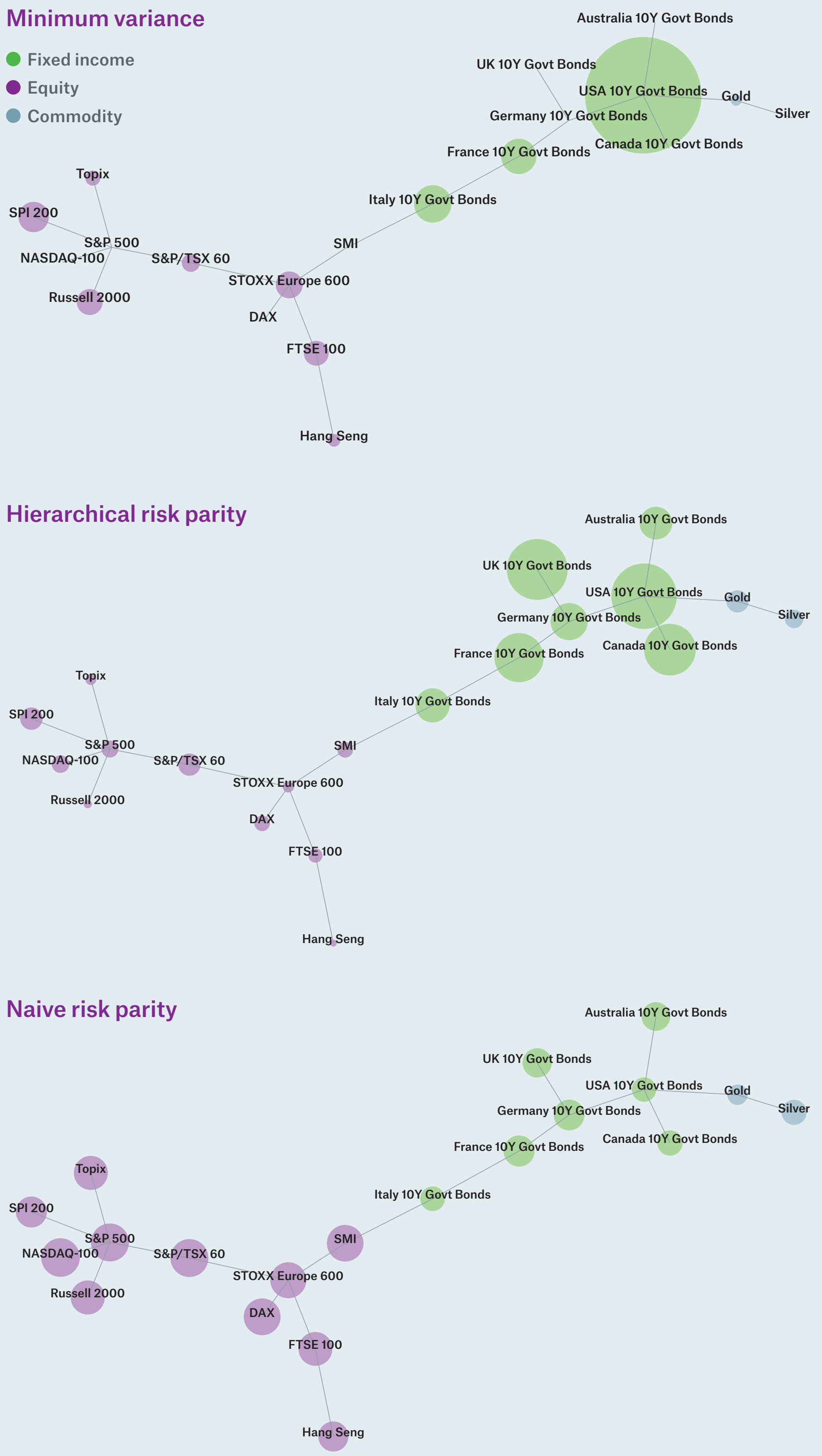

Figure 9

While minimum variance often leads to assign large amounts of risk weight to individual assets, naive risk parity is more evenly distributing – thus disregarding potential clusters of assets. HRP sits in the well-balanced middle of those two extremes.

Source: Munich Re

We notice that HRP shows a balanced distribution of risk over assets and asset classes. At first glance, the weights of the inverse volatility portfolio do look smoothly distributed as well but looking at the risk arising from the different asset classes it becomes visible that a large amount of risks is allocated to equities. On average, approximately 60% of the risk originates from equities. For the equally-weighted risk contribution (ERC) approach, the risk contribution per asset is the same by definition. The risk per asset class therefore is simply proportional to the number of assets within the asset class, i.e. 55% of the risk arises from equities. For HRP and minimum variance on average 30% of the risk stems from equities.

Let's have a look at the concentration of the portfolios and measure how good the diversification across the different portfolios and allocation methods is. In general, there are various metrics that are used to measure the diversification of a portfolio.

Here, we focus on the commonly used metrics maximum risk contribution, diversification ratio, concentration ratio and the number of uncorrelated exposures.

Most of the allocation methods are specifically designed to optimise a single concentration measure and hence by definition might look very favourable for this metric. For example, EW minimises the maximum weight, ERC minimises the maximum risk contribution, the most-diversified portfolio (MDP, not considered in this article) optimises the diversification ratio and the concentration ratio is minimised by NRP.

Concentration metrics

Maximum risk contribution

The metric shows the highest amount of risk attributed to one individual asset during the time window under consideration.

Diversification ratio

The diversification ratio (DR) measures the volatility reduction in a portfolio originating from diversification effects (Choueifaty, 2008)

$$\mathrm{DR} = \frac{\sum \omega_i \sigma_i}{\sqrt{\omega' \Sigma \omega}} .$$

Concentration ratio

The diversification ratio can be decomposed into a volatility-weighted average correlation and a volatility-weighted concentration ratio (CR)

$$\mathrm{CR} = \frac{\sum (\omega_i \sigma_i)^2}{(\sum \omega_i \sigma_i)^2} .$$

The decomposition $\mathrm{DR} = (\rho(1 - \mathrm{CR}) + \mathrm{CR})^{-\frac{1}{2}}$ shows that for a given portfolio the diversification ratio can be increased by an decreasing concentration ratio.

Uncorrelated exposures

The number of uncorrelated exposure was introduced by Meucci (2009). Deduced from a principal component analysis (PCA) the portfolio is decomposed into a number of uncorrelated sources of risk (Principal Portfolios). The metric can be defined as the exponential of the negative Shannon entropy
$\mathrm{N} = \exp(-\sum p_i \log p_i)$.

Figure 10
Source: Munich Re

Concentration metrics [7, 8]

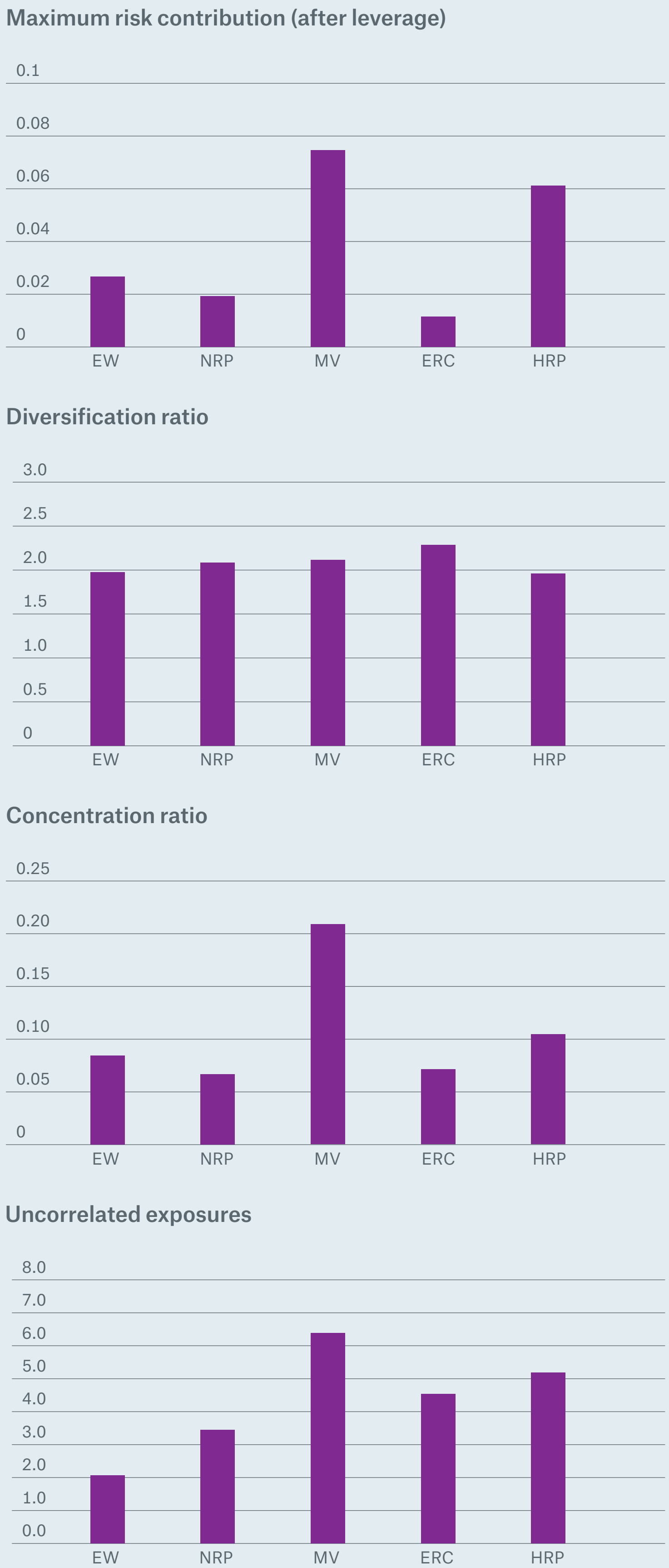

Figure 11

HRP together with ERC show the most balanced concentration metrics. Especially the low concentration ratio in combination with a high number of uncorrelated exposures do look very favorable for the HRP approach.

Source: Munich Re

[7] The metrics diversification ratio, concentration ratio and number of uncorrelated exposures were calculated on each monthly rebalancing date and afterwards we formed the average of those values.

[8] Please note that some constituents were not available at the start of the index and therefore the presented diversification of the portfolios tends to be lower than what is observed in recent years.

MV and HRP show high maximum risk contributions stemming from high allocations in the fixed income segment but also due to the higher leverage. The diversification ratios are on a similar level. MV shows by far the highest concentration ratio but also the highest amount of uncorrelated exposures.

For HRP, we notice that it looks well-balanced between a proper concentration and a high number of uncorrelated exposures. This holds also for the ERC method, which yields a very positive index performance but mainly suffered from worse drawdowns in the years following the Dotcom Bubble and the Global Financial Crisis.

Quintessence

The efficient and optimal composition of investment portfolios is a key challenge for all kind of investors. For more than half a century, the interpretation of the term "optimal" has been dominated by MPT-style approaches to asset allocation. While Markowitz (1952) without any doubt represents one of the most important milestones in finance, its theoretical elegance is often not mirrored in reality .

Originating from graph theory and machine learning, the HRP approach provides a new contemporary prescription to the traditional challenges of asset allocation. A key aspect is the introduction of hierarchical relationships amongst portfolio components. This has an important consequence: asset weight budget is no longer able to fluctuate freely and find unintended ways in a fully connected asset universe. As a consequence, HRP enforces more stable solutions, which are also more in line with intuition.

The HRP approach manages to benefit from stable diversification effects and a well-balanced allocation between different asset classes, both in theory and in practice. It incorporates the correlation structure of the portfolio in its straightforward and coherent construction principle and at the same time avoids the problems of the inversion of the covariance matrix.

Especially during extreme and challenging market phases, the HRP method can profit from its stable and diversified allocation and it is outstanding in minimising losses. This mitigation of steep drawdowns is key for the remarkable long-term outperformance of other major allocation principles.

Given the remarkably stable results and low drawdowns, HRP appears to be an ideal candidate for long-term investing in the context of life insurance and pension schemes.

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